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Semion Gutman[1] and Alexander G Ramm[2]

[1] Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA
[2] Department of Mathematics, Kansas State University, Manhattan, KS 66506-2602, USA

E-mail: sgutman@ou.edu and ramm@math.ksu.edu

Abstract
The goal of this work is to show that the numerical solution of the obstacle
scattering problem based on the modified Rayleigh conjecture (MRC) method
is a competitive alternative to the boundary integral equations method, and
that it has numerical advantages which may be especially important in three-
dimensional scattering problems with non-smooth domains, for example, with
domains whose boundaries contain corners. The MRC is formulated, the
algorithm based on it is described and numerical results are presented.

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1. Introduction

The goal of this work is to show that the numerical solution of the obstacle scattering problem
based on the modified Rayleigh conjecture (MRC) method is a competitive alternative to the
boundary integral equations method (BIEM), and that it has numerical advantages which may
be especially important in three-dimensional scattering problems with non-smooth domains,
for example, with domains whose boundaries contain corners. Several numerical examples
illustrate the above conclusions. In section 1 the statement of the problem is formulated in
the two-dimensional case. In section 2 the MRC is formulated (see [8]), and its extension
(theorem 2.2) convenient for the numerical realization is given. In section 3 an algorithm,
based on MRC, for solving the direct scattering problem is given. In section 4 the results of
the numerical experiments are presented.

We start with the formulation of the obstacle scattering problem. In this paper, we consider
the Dirichlet boundary condition, but the method we develop can be used for the Neumann
and Robin boundary conditions.

Let an obstacle be a bounded domain $D \subset \mathbb{R}^2$ with a Lipschitz boundary Γ. Fix a
frequency $k > 0$ and denote the exterior domain by $D' = \mathbb{R}^2 \setminus \bar{D}$.

A solution $v(x)$ of the Helmholtz equation

$$\Delta v + k^2 v = 0 \qquad x \in D' \tag{1.1}$$

is called outgoing if it satisfies the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} \sqrt{|x|} \left(\frac{\partial v}{\partial |x|} - \mathrm{i}kv \right) = 0 \tag{1.2}$$

where the limit is attained uniformly for all the directions $x/|x|$, $x \in \mathbb{R}^2$.

The *exterior Dirichlet problem* consists in finding an outgoing solution of the Helmholtz equation (1.1) which satisfies the boundary condition

$$v = f \qquad x \in \Gamma \tag{1.3}$$

where f is a continuous function, see [6] for the existence and uniqueness results for this problem.

A particular case of the above problem is the *direct acoustic obstacle scattering problem*. Let $\alpha \in S^1$, and the incident field be

$$u^i(x) = \mathrm{e}^{\mathrm{i}kx\cdot\alpha}. \tag{1.4}$$

The problem is to find the total field

$$u(x,k) = u^i + u^s \qquad x \in D' \tag{1.5}$$

such that

$$u = 0 \qquad x \in \Gamma \tag{1.6}$$

and the scattered field u^s satisfies (1.1) and (1.2).

It is known (see e.g., [6]), that every outgoing solution $v(x)$, $x \in D'$ has an asymptotic representation

$$v(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{\sqrt{|x|}} \left\{ A(\alpha') + O\left(\frac{1}{|x|}\right) \right\} \qquad |x| \to \infty \tag{1.7}$$

where $\alpha' = x/|x|$, $\alpha' \in S^1$. The function $A(\alpha') := A_v(\alpha')$ is called the far field pattern of u.

For the direct acoustic obstacle scattering problem this representation takes the form

$$u^s(x) = \frac{\mathrm{e}^{\mathrm{i}k|x|}}{\sqrt{|x|}} \left\{ A(\alpha',\alpha) + O\left(\frac{1}{|x|}\right) \right\} \qquad |x| \to \infty \tag{1.8}$$

where the uniquely defined function $A(\alpha',\alpha)$ is called the scattering amplitude of the obstacle scattering problem.

Let $J_l(t)$ and $Y_l(t)$ be the Bessel and Neumann functions of the integer order l. The first Hankel function of order l is defined by $H_l^{(1)} = J_l + \mathrm{i}N_l$. Suppose that the circle $B_R = \{x \in \mathbb{R}^2 : |x| \leqslant R\}$ contains D. Then, in the region $|x| > R$, the outgoing solution of the exterior Dirichlet problem (1.1)–(1.3) has a unique representation

$$v(x) = \sum_{l=-\infty}^{\infty} a_l H_l^{(1)}(k|x|)\, \mathrm{e}^{\mathrm{i}l\theta} \tag{1.9}$$

where $x/|x| = (\cos(\theta), \sin(\theta))$.

The Rayleigh conjecture (RC) states that the series (1.9) converges up to the boundary Γ. This conjecture is false, see [1, 4, 6]. Recently Ramm [8] established a modified Rayleigh conjecture (MRC). In this paper, the MRC approach is demonstrated to be a viable alternative for computational direct scattering problems. In [8] a method for solving the inverse obstacle scattering problem is also proposed. We plan to work on its numerical implementation in the near future. In [7] multidimensional inverse scattering problems are studied.

2. Modified Rayleigh conjecture (MRC)

A 2D version of the main result from [8] is

Theorem 2.1. *Let (1.9) be the unique representation of the outgoing solution $v(x)$ of the exterior Dirichlet problem (1.1)–(1.3). Fix an $\epsilon > 0$.*

Then there exists a positive integer $L = L(\epsilon)$ and the coefficients $c_l = c_l(\epsilon), l = 0, \pm 1, \ldots, \pm L$ such that

(i)

$$\left\| f(x) - \sum_{l=-L}^{L} c_l H_l^{(1)}(k|x|) \, e^{il\theta} \right\|_{L^2(\Gamma)} \leqslant \epsilon$$

(ii)

$$\left\| v(x) - \sum_{l=-L}^{L} c_l H_l^{(1)}(k|x|) \, e^{il\theta} \right\| = O(\epsilon) \qquad \epsilon \to 0$$

where

$$\| \cdot \| = \| \cdot \|_{H_{loc}^m(D')} + \| \cdot \|_{L^2(D';(1+|x|)^{-\gamma})}$$

$\gamma > 1, m > 0$ is an arbitrary integer, H^m is the Sobolev space and

(iii)

$$c_l(\epsilon) \to a_l \qquad as \qquad \epsilon \to 0 \qquad l = 0, \pm 1, \pm 2, \ldots.$$

Proof. (i) Without loss of generality we can assume that the origin is an interior point of the domain D. Therefore there exists $\delta > 0$ such that $B_\delta = \{x \in \mathbb{R}^2 : |x| < \delta\} \cap \Gamma = \emptyset$, where Γ is the boundary of D. Suppose that a parametric equation of Γ is

$$\mathbf{r}(t) = \langle x_1(t), x_2(t) \rangle \qquad 0 \leqslant t < 2\pi. \tag{2.1}$$

Let $r(t) = |\mathbf{r}(t)|$. Following [6], it is enough to show that functions

$$\psi_l(t) = H_l^{(1)}(k|x|) \, e^{il\theta}\big|_{\Gamma} = H_l^{(1)}(k|r(t)|) \, e^{il\theta(t)} \qquad 0 \leqslant t < 2\pi \tag{2.2}$$

where $e^{i\theta(t)} = \mathbf{r}(t)/r(t)$, and $l = 0, \pm 1, \pm 2 \ldots$, form a complete system in $L^2(\Gamma)$. Here and in the following unit vectors in $\mathbb{R}^2$ and the corresponding complex numbers are identified as needed.

Suppose that there exists $g \in L^2(\Gamma)$ such that

$$\int_0^{2\pi} g(t)\psi_l(t) \, dt = 0 \qquad l = 0, \pm 1, \pm 2, \ldots. \tag{2.3}$$

By the addition theorem for any $x, y \in \mathbb{R}^2$ with $|x| > |y|$ we have

$$H_0^{(1)}(k|x - y|) = \sum_{l=-\infty}^{\infty} H_l^{(1)}(k|x|) \, e^{il\theta_x} J_l(k|y|) \, e^{-il\theta_y} \tag{2.4}$$

where $x/|x| = e^{i\theta_x}$ and $y/|y| = e^{i\theta_y}$.

Recalling that the fundamental solution to the Helmholtz equation in two dimensions is

$$\Phi(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|) \qquad x \neq y$$

define the single-layer potential of g by

$$w(y) = \frac{\mathrm{i}}{4} \int_\Gamma H_0^{(1)}(k|x-y|) g(x) \, \mathrm{d}s(x) \tag{2.5}$$

$$= \frac{\mathrm{i}}{4} \sum_{l=-\infty}^{\infty} J_l(k|y|) \, \mathrm{e}^{-\mathrm{i}l\theta_y} \int_0^{2\pi} g(t)\psi_l(t) \, \mathrm{d}t \qquad y \in B_\delta. \tag{2.6}$$

Thus $w(y) = 0$ for any $y \in B_\delta$. By the unique continuation property, $w(y)$ is identically equal to zero in D. The continuity of the single-layer potential implies that $w(y) = 0$, $y \in \Gamma$. Since $w(y)$ is an outgoing solution of the Helmholtz equation in D', vanishing on Γ, it follows that $w(y) = 0$, $y \in D'$. Finally, the jump properties of the normal derivative of the single-layer potential imply that $g = 0$ in $L^2(\Gamma)$.

(ii) Let the coefficients $c_l = c_l(\epsilon)$ be chosen according to (i). Denote

$$w_\epsilon(x) = v(x) - \sum_{l=-L}^{L} c_l H_l^{(1)}(k|x|) \, \mathrm{e}^{\mathrm{i}l\theta}. \tag{2.7}$$

According to Green's formula

$$w_\epsilon(x) = -\int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu(y)} w_\epsilon(y) \, \mathrm{d}s(y) \qquad x \in D'. \tag{2.8}$$

Let $D \subset B_R = \{x \in \mathbb{R}^2 : |x| < R\}$, and S_R be the boundary of B_R. Since

$$\|w_\epsilon\|_{L^2(\Gamma)} < \epsilon \tag{2.9}$$

and

$$\left| \frac{\partial \Phi(x,y)}{\partial \nu(y)} \right| \leqslant \frac{c}{1 + \sqrt{|x|}} \qquad |x| \geqslant R$$

it follows that

$$\|w_\epsilon\|_{L^2(S_R)} < c\epsilon \tag{2.10}$$

and

$$|w_\epsilon(x)| < c\epsilon(1 + \sqrt{|x|})^{-1} \qquad |x| \geqslant R. \tag{2.11}$$

Therefore, w_ϵ is $O(\epsilon)$ in the norm $L^2(D'; (1 + \sqrt{|x|})^{-\gamma})$ for $\gamma > 1$.

Let $D'_R = B_R \setminus D$. Choose R such that k^2 is not a Dirichlet eigenvalue of $-\Delta$ in D'_R. Then w_ϵ satisfies the following elliptic estimate from ([3], p 189):

$$\|w_\epsilon\|_{H^m(D'_R)} \leqslant c \left[\|(\Delta + k^2)w_\epsilon\|_{H^{m-2}(D'_R)} + \|w_\epsilon\|_{H^{m-1/2}(S_R)} + \|w_\epsilon\|_{H^{m-1/2}(\Gamma)} \right]. \tag{2.12}$$

Since $(\Delta + k^2)w_\epsilon = 0$ in D' this formula with $m = 1/2$ together with (2.9) and (2.10) gives

$$\|w_\epsilon\|_{H^{1/2}(D'_R)} \leqslant c\epsilon. \tag{2.13}$$

From local elliptic estimates and (2.13) it follows that (ii) of theorem 2.1 holds with any m.

(iii) Let $x \in S_R$. By (2.7)

$$w_\epsilon(x) = \sum_{l=-L}^{L} (a_l - c_l) H_l^{(1)}(kR) \, \mathrm{e}^{\mathrm{i}l\theta} + \sum_{l>|L|} a_l H_l^{(1)}(kR) \, \mathrm{e}^{\mathrm{i}l\theta}. \tag{2.14}$$

Bessel's inequality and (2.10) imply

$$|a_l - c_l| \leqslant \|w_\epsilon\|_{L^2(S_R)} < c\epsilon$$

and (iii) follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

According to theorem 2.1 the computation of the outgoing solution of (1.1)–(1.3) is reduced to the approximation of the boundary values in (1.3) by functions $\psi_l(t)$. A direct implementation of the above algorithm is efficient for domains D not very different from a

circle, e.g. for an ellipse with a small eccentricity (see numerical experiments in the next section). The numerical difficulties happen because the Neumann functions N_l with large values of l are bigger than N_l with small values of l by many orders of magnitude. A finite precision of numerical computations makes it necessary to keep the values of L not high, e.g. $L \leqslant 20$. This restriction can be remedied by adding to the set $\{\psi_l\}$ similar functions, centred at other interior points $x_j \in D$. Such an approach is justified by the following theorem:

Theorem 2.2. *Let $v(x)$ be the outgoing solution of the exterior Dirichlet problem (1.1)–(1.3). Suppose that points $x_1, x_2, \ldots, x_J$ are in the interior of D, and $\epsilon > 0$.*

   Then

(i) *there exists a positive integer $L = L(\epsilon)$ and the coefficients $c_{lj} = c_{lj}(\epsilon), l = 0, \pm 1, \ldots, \pm L, j = 1, 2, \ldots, J$ such that*

$$\left\| f(x) - \sum_{j=1}^{J} \sum_{l=-L}^{L} c_{lj} H_l^{(1)}(k|x - x_j|) \, e^{il\theta_j} \right\|_{L^2(\Gamma)} \leqslant \epsilon \qquad (2.15)$$

   where $(x - x_j)/|x - x_j| = e^{i\theta_j}$.

(ii) *Let*

$$v_\epsilon(x) = \sum_{j=1}^{J} \sum_{l=-L}^{L} c_{lj} H_l^{(1)}(k|x - x_j|) \, e^{il\theta_j} \qquad (2.16)$$

   then

$$\|v(x) - v_\epsilon(x)\| = O(\epsilon) \qquad \epsilon \to 0$$

   where

$$\| \cdot \| = \| \cdot \|_{H_{loc}^m(D')} + \| \cdot \|_{L^2(D';(1+|x|)^{-\gamma})}$$

   $\gamma > 1, m > 0$ is an arbitrary integer, H^m is the Sobolev space.

(iii) *The far field pattern of the approximate solution $v_\epsilon(x)$ is given by*

$$A_{v_\epsilon}(\alpha') = \sqrt{\frac{2}{\pi k}} \, e^{-i\frac{\pi}{4}} \sum_{j=1}^{J} \left(e^{-ik\alpha' \cdot x_j} \sum_{l=-L}^{L} c_{lj}(-i)^l \, e^{il\theta} \right) \qquad (2.17)$$

   where $\alpha' = x/|x| = e^{i\theta}$.

Proof. Items (i) and (ii) follow from theorem 2.1. For item (iii) note that

$$H_n^{(1)}(t) = \sqrt{\frac{2}{\pi t}} \, e^{i(t - \pi n/2 - \pi/4)} \left\{ 1 + O\left(\frac{1}{t}\right) \right\} \qquad t \to \infty$$

and

$$|x - x_j| = |x| - \alpha' \cdot x_j + O\left(\frac{1}{|x|}\right) \qquad |x| \to \infty.$$

Therefore,

$$H_l^{(1)}(k|x - x_j|) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ \sqrt{\frac{2}{\pi k}} \, e^{i(-k\alpha' \cdot x_j - \pi l/2 - \pi/4)} + O\left(\frac{1}{|x|}\right) \right\} \qquad |x| \to \infty.$$

Finally,

$$e^{i\theta_j} = \frac{x - x_j}{|x - x_j|} = \alpha' + O\left(\frac{1}{|x|}\right) \qquad |x| \to \infty. \qquad \square$$

3. Direct scattering problem via MRC

According to theorem 2.2 one can approximate the scattered field u^s of the direct obstacle scattering problem (1.4)–(1.6) by minimizing (2.15) with $f(x) = -u^i(x), x \in \Gamma$. More

precisely, the algorithm can be described as follows:

Initialization. Fix an integer $L > 0$ and an $\epsilon > 0$. Choose $x_1, x_2, \ldots, x_J$ in the interior of D. If $\mathbf{r}(t), 0 \leqslant t < 2\pi$ is an equation of the boundary Γ, let

$$\psi_{lj}(t) = H_l^{(1)}(k|\mathbf{r}(t) - x_j|)\, \mathrm{e}^{\mathrm{i}l\theta_j(t)} \qquad j = 1, 2, \ldots, J \qquad l = 0, \pm 1, \pm 2, \ldots, \pm L \quad (3.1)$$

$$g(t) = -u^i(\mathbf{r}(t)) = -\mathrm{e}^{\mathrm{i}k\mathbf{r}(t)\cdot\alpha} \tag{3.2}$$

where $(\mathbf{r}(t) - x_j)/|\mathbf{r}(t) - x_j| = \mathrm{e}^{\mathrm{i}\theta_j(t)}$.

Minimization. Minimize

$$\Phi(\mathbf{c}) = \left\| g(t) - \sum_{j=1}^{J} \sum_{l=-L}^{L} c_{lj}\psi_{lj}(t) \right\|_{L^2(0,2\pi)} \qquad \mathbf{c} = \{c_{lj}\} \tag{3.3}$$

for $\mathbf{c} \in \mathbb{C}^N$, $N = (2L + 1)J$.

If the minimum of Φ in (3.3) is smaller than the prescribed tolerance ϵ, then the scattered field is approximated by $v_\epsilon(x), x \in D'$, given by (2.16), and its scattering amplitude $A(\alpha', \alpha)$ is computed by formula (2.17).

Numerical experiments also show that it may be beneficial to complement the set of functions ψ_{lj} by other known outgoing solutions in D'.

An example of such a solution $v(x)$ is the field scattered by a sphere $B_a(x_0) \subset D$ of radius $a > 0$, subjected to the same incident wave $u^i(x) = \mathrm{e}^{\mathrm{i}k\alpha\cdot x}$ as in the original direct scattering problem. It is given by

$$v(x) = -\mathrm{e}^{\mathrm{i}k\alpha\cdot x_0} \sum_{l=-\infty}^{\infty} \mathrm{i}^l \frac{J_l(ka)}{H_l^{(1)}(ka)} H_l^{(1)}(k|x - x_0|)\, \mathrm{e}^{\mathrm{i}l(\theta_0-\beta)} \tag{3.4}$$

and, because $\theta_0 \to \theta$ as $|x| \to \infty$, $x/|x| = \alpha'$, its scattering amplitude is

$$A_v(\alpha', \alpha) = -\sqrt{\frac{2}{\pi k}}\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}\, \mathrm{e}^{\mathrm{i}k(\alpha-\alpha')\cdot x_0} \sum_{l=-\infty}^{\infty} \frac{J_l(ka)}{H_l^{(1)}(ka)}\, \mathrm{e}^{\mathrm{i}l(\theta-\beta)} \tag{3.5}$$

where $\alpha' = x/|x| = \mathrm{e}^{\mathrm{i}\theta}$, $\alpha = \mathrm{e}^{\mathrm{i}\beta}$ and $(x - x_0)/|x - x_0| = \mathrm{e}^{\mathrm{i}\theta_0}$.

The numerical implementation of the minimization algorithm begins with the choice of M knots $0 = t_1 < t_2 < \cdots < t_M < 2\pi$, and points x_j, $j = 1, \ldots, J$ in the interior of D. Then the values $\{\psi_{lj}(t_m)\}_{m=1}^{M}$ form $N = (2L + 1)J$ vectors $\mathbf{a}^{(n)}$, $n = 1, 2, \ldots, N$ of length M. Let $\mathbf{b} = \{-u^i(t_m)\}_{m=1}^{M}$. Then the minimization problem (3.3) is reduced to the finite-dimensional minimization problem

$$\min\{\|A\mathbf{c} - \mathbf{b}\|, \mathbf{c} \in \mathbb{C}^N\} \tag{3.6}$$

where A is the matrix containing vectors $\mathbf{a}^{(n)}$, $n = 1, 2, \ldots, N$ as its columns. If other outgoing solutions are used in addition to functions ψ_{lj}, the size of matrix A is increased accordingly.

We use the singular value decomposition (SVD) method (see e.g., [5]) to minimize (3.6). Small singular values of the matrix A are used to identify and delete linearly dependent or almost linearly dependent combinations of vectors $\mathbf{a}^{(n)}$. This spectral cut-off makes the minimization process stable. The entire algorithm is summarized below. We denote by V^H the complex conjugate transpose of a matrix V. Also, by definition, the inner product in $\mathbb{C}^N$ complex conjugates its first component.

Iterative MRC. Fix an $\epsilon > 0$, an integer $L > 0$, and $w_{\min} > 0$.

Choose M knots $0 = t_1 < t_2 < \cdots < t_M < 2\pi$, and points x_j, $j = 1, \ldots, J$ in the interior of D.

Let $N = (2L + 1)J$.

(1) **Initialization**

    (a) Form vectors

$$\mathbf{a}^{(n)} = \{\psi_{lj}(t_m)\}_{m=1}^{M} \qquad |l| \leqslant L \qquad j = 1, 2, \ldots, J$$

    and the matrix A of size $M \times N$, whose columns are the vectors $\mathbf{a}^{(n)}$.

    (b) Form vector

$$\mathbf{b} = \{-u^i(t_m)\}_{m=1}^{M}.$$

    (c) Use singular value decomposition method to represent matrix A as

$$A = UWV^H$$

    where the $M \times N$ matrix U has orthonormal columns $\mathbf{u}^{(n)}$, $n = 1, \ldots, N$, the square $N \times N$ matrix V has orthonormal columns $\mathbf{v}^{(n)}$, $n = 1, \ldots, N$ and the diagonal square $N \times N$ matrix $W = (w_n)_{n=1}^{N}$ is composed of the (nonnegative) singular values of A.

    (d) Let $\Sigma \subset \{1, 2, \ldots, N\}$ be defined by

$$\Sigma = \{n : w_n \geqslant w_{\min}\}.$$

    (e) Let $p = 0$.

(2) **Iterative step**

    (a) Let $p := p + 1$.

    (b) Form the set

$$P = \{n \in \Sigma : w_n \text{ is among } p \text{ largest singular values of } A\}.$$

    (c) Compute the normalized residual

$$r_p^{\min} = \frac{1}{\sqrt{M}} \sqrt{\|\mathbf{b}\|^2 - \sum_{n \in P} |\langle \mathbf{u}^{(n)}, \mathbf{b} \rangle|^2}.$$

(3) **Stopping criterion**

    (a) If $r_p^{\min} \leqslant \epsilon$, then stop.
    The minimizer is given by

$$\mathbf{c} = \sum_{n \in P} \frac{1}{w_n} \langle \mathbf{u}^{(n)}, \mathbf{b} \rangle \mathbf{v}^{(n)}.$$

    Compute the scattered field v_ϵ^s using (2.16) and the far field pattern using (2.17).

    (b) If $r_p^{\min} > \epsilon$, and $P \neq \Sigma$ repeat the iterative step (2).

    (c) Otherwise, the procedure has failed.

4. Numerical experiments

The results obtained by the MRC method (for smooth boundary Γ) were compared to the results obtained by the boundary integral equation method (BIEM) as presented in [2]. Accordingly, to find the outgoing solution $v(x)$ of the exterior Dirichlet problem (1.1)–(1.3) one has to solve the integral equation

$$\varphi(x) + \int_\Gamma \left\{ \frac{\partial \Phi(x, y)}{\partial v(y)} - i\eta \Phi(x, y) \right\} \varphi(y) \, ds(y) = 2f(x) \qquad x \in D' \quad (4.1)$$

for the density function $\varphi \in C(\Gamma)$. Following the recommendations in [2] the value for the real coupling parameter η was chosen to be equal to the wave number k. The above integral equation was solved using the Nyström method, see [2], section 3.5.

Table 1. Normalized residuals attained in the numerical experiments.

Experiment	J	k	α	$r^{\min}$	$\hat{r}^{\min}$
I	4	1.0	(1.0, 0.0)	0.000 201	0.000 096
	4	1.0	(0.0, 1.0)	0.000 357	0.000 324
	4	5.0	(1.0, 0.0)	0.001 309	0.000 556
	4	5.0	(0.0, 1.0)	0.007 228	0.002 156
II	16	1.0	(1.0, 0.0)	0.003 555	0.003 555
	16	1.0	(0.0, 1.0)	0.002 169	0.002 169
	16	5.0	(1.0, 0.0)	0.009 673	0.009 286
	16	5.0	(0.0, 1.0)	0.007 291	0.007 056
III	16	1.0	(1.0, 0.0)	0.008 281	0.008 281
	16	1.0	(0.0, 1.0)	0.007 523	0.007 523
	16	5.0	(1.0, 0.0)	0.021 571	0.021 530
	16	5.0	(0.0, 1.0)	0.024 360	0.024 206
IV	32	1.0	(1.0, 0.0)	0.006 610	0.006 610
	32	1.0	(0.0, 1.0)	0.006 785	0.006 785
	32	5.0	(1.0, 0.0)	0.034 027	0.033 218
	32	5.0	(0.0, 1.0)	0.040 129	0.038 558

After the density φ is computed, the far field pattern can be obtained from

$$A_v(\alpha') = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_\Gamma \{k\nu(y) \cdot \alpha' + \eta\} \, e^{-ik\alpha' \cdot y} \varphi(y) \, ds(y) \tag{4.2}$$

where ν is the exterior unit normal vector to the boundary Γ.

We conducted numerical experiments for four obstacles: two ellipses of different eccentricity, a kite and a triangle. Each case was tested for wave numbers $k = 1.0$ and $k = 5.0$. Each obstacle was subjected to incident waves corresponding to $\alpha = (1.0, 0.0)$ and $\alpha = (0.0, 1.0)$. The results are shown in table 1. The column J shows the number of the interior points x_j used in the approximation (3.3). The choice of the points x_j was different in each experiment. It is indicated below together with the description of the experiments. The column $r^{\min}$ shows the smallest value of the normalized residual achieved by the MRC minimization method in step 2(c) of the iterative MRC algorithm described in the previous section.

We also tested the algorithm with the outgoing solutions (3.4) induced by spheres in D. These scattering solutions were used in addition to functions ψ_{lj}. The spheres were centred at the same interior points x_j chosen in the experiment. The radius of each of the spheres is chosen to be the biggest one that fits it within the domain D. Thus we allow the spheres to intersect each other, but still be entirely inside the domain D. The column with $\hat{r}^{\min}$ shows the values of the normalized residual attained in this version of the MRC algorithm. Values $L = 5$ and $M = 720$ were used in all the experiments. The 720 knots t_m were uniformly distributed on $[0, 2\pi]$.

Table 2 shows the scattering amplitude for vectors α' at every multiple of 30° for experiment I, and the ratio (denoted by MRC/BIEM) of the scattering amplitude computed by the MRC method and by the BIEM. The BIEM was implemented as described in the beginning of this section with $n = 64$ (see [2]). The scattering amplitudes shown correspond to the last column in table 1.

Experiment I. The boundary Γ is an ellipse described by

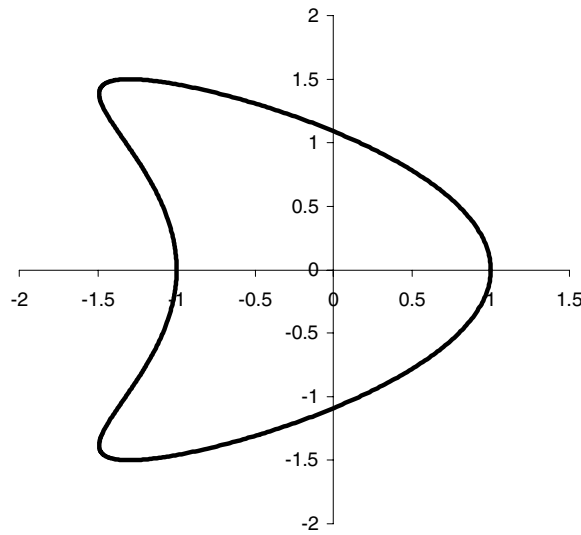$$\mathbf{r}(t) = (2.0\cos t, \sin t) \qquad 0 \leqslant t < 2\pi. \tag{4.3}$$

Figure 1. The kite-shaped obstacle used in experiment II.

Table 2. Scattering amplitude $A(\alpha', \alpha)$ in experiment I for $\alpha = (1.0, 0.0)$ and $k = 5.0$.

α'	$A(\alpha', \alpha)$	MRC/BIEM
0°	$-0.016258 - 0.642831i$	$1.000115 + 0.000082i$
30°	$0.413163 + 0.337149i$	$1.000213 - 0.000035i$
60°	$0.273417 - 0.621033i$	$0.999987 + 0.000300i$
90°	$-3.000307 + 2.365251i$	$0.999990 - 0.000084i$
120°	$0.273418 - 0.621033i$	$0.999912 + 0.000406i$
150°	$0.413163 + 0.337149i$	$1.000135 - 0.000022i$
180°	$-0.016258 - 0.642831i$	$1.000001 - 0.000001i$
210°	$-0.341318 - 0.799098i$	$1.000041 - 0.000014i$
240°	$0.482573 - 1.109062i$	$0.999997 - 0.000034i$
270°	$1.176199 - 0.783961i$	$0.999960 - 0.000041i$
300°	$0.482572 - 1.109062i$	$1.000039 - 0.000093i$
330°	$-0.341318 - 0.799098i$	$1.000002 + 0.000016i$

The MRC minimization used four interior points $x_j = 0.7\mathbf{r}\left(\frac{\pi(j-1)}{2}\right)$, $j = 1, \ldots, 4$. Run time for the MRC was 2 s versus 25 s for the BIEM on a 333 MHz PC.

Experiment II. The kite-shaped boundary Γ (see [2], section 3.5) is described by

$$\mathbf{r}(t) = (-0.65 + \cos t + 0.65 \cos 2t, \, 1.5 \sin t) \qquad 0 \leqslant t < 2\pi \qquad (4.4)$$

see figure 1. The MRC minimization used 16 interior points $x_j = 0.9\mathbf{r}\left(\frac{\pi(j-1)}{8}\right)$, $j = 1, \ldots, 16$. Run time for the MRC was 33 s versus 44 s for the BIEM.

Experiment III. The boundary Γ is the triangle with vertices at $(-1.0, 0.0)$ and $(1.0, \pm 1.0)$. The MRC minimization used 16 interior points $x_j = 0.9\mathbf{r}\left(\frac{\pi(j-1)}{8}\right)$, $j = 1, \ldots, 16$. Run time for the MRC was about 30 s. See figure 2 for the scattering amplitude $A(\alpha', \alpha)$ for $\alpha = (0.0, 1.0)$, $k = 5.0$ obtained by the MRC method with these parameters.

Experiment IV. The boundary Γ is an ellipse described by

$$\mathbf{r}(t) = (0.1 \cos t, \, \sin t) \qquad 0 \leqslant t < 2\pi. \qquad (4.5)$$
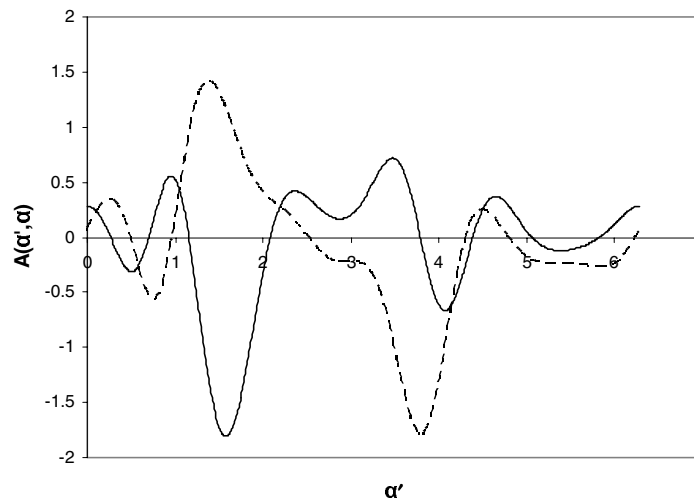
Figure 2. Real (solid line) and imaginary (dotted line) parts of the scattering amplitude $A(\alpha', \alpha)$ for experiment III, $\alpha = (0.0, 1.0)$ and $k = 5.0$.

The MRC minimization used 32 interior points $x_j = 0.95\mathbf{r}\left(\frac{\pi(j-1)}{16}\right)$, $j = 1, \ldots, 32$. Run time for the MRC was about 140 s.

5. Conclusions

Computation of scattering solutions via the modified Rayleigh conjecture method provides an alternative to the boundary integral equation method. BIEM representations (4.1) and (4.2) are exact, while MRC provides an approximate method. The advantages of MRC consist in its easy implementation in both 2D and 3D cases. Also its performance is shown to be better, or, at least, equal to the BIEM. In contrast to BIEM, no additional programming is needed for domains with corners. Additional studies are needed to fine-tune the performance of the MRC algorithm, but it seems that the most dramatic improvement over the BIEM may be for 3D obstacles. The usage of the outgoing waves, scattered by spheres, improves the algorithm, especially if their centres are located away from the boundary of the obstacle. The improvement is more pronounced for values of $L < 5$. The choice of the spheres inscribed in the domain D was the most efficient in the conducted experiments.

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